TRINITY COLLEGE

IN NOMINE DOMINI

Semester Two Examination, 2018

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two:
Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________________

Your name ______________________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	51	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(6 marks)**

The level of Strontium-90 in a contaminated soil sample at the start of 1985 was 0.65 mg/kg. Strontium-90 has a half-life of 28.2 years and decays continuously such that $S = S_0 e^{kt}$ where S is the level of Strontium-90, t is the time in years since the level was S_0 and k is a constant.

(a) Assuming no further contamination occurred, determine

(i) the level of Strontium-90 in the sample at the start of 2018. (3 marks)

Solution
$0.5 = e^{28.2k}$ $k = -0.02458$ $S(33) = 0.65e^{-0.02458(33)} = 0.289\ \text{mg/kg}$
Specific behaviours
✓ writes equation for k ✓ value of k ✓ value for S that rounds to 0.29

(ii) the rate of change of the level of Strontium-90 in the sample at the start of 2018. (1 mark)

Solution
$-0.02458 \times 0.289 = -0.0071\ \text{mg/kg/year}$
Specific behaviours
✓ answer (i) multiplied by k

(b) Strontium-90 decays into Yttrium-90. The mass of Yttrium-90 decays continuously such that $Y = Y_0 e^{-0.0112t}$ where Y is the mass of Yttrium-90 and t is the time in hours since the level was Y_0. Determine the time taken for a mass of Yttrium-90 to decrease by 80%. (2 marks)

Solution
$e^{-0.0112t} = 0.20$ $t = 144\ \text{hours}$
Specific behaviours
✓ writes equation for t ✓ solves for t

Question 10 **(8 marks)**

The discrete random variable X has $\mathrm{E}(X) = 3.2$ and probability function

$$P(X = x) = \begin{cases} a + bx & x = 2, 3, 4 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine the values of the constants a and b. (4 marks)

Solution

x	2	3	4
$P(X = x)$	$a + 2b$	$a + 3b$	$a + 4b$

Sum of probabilities: $a + 2b + a + 3b + a + 4b = 1$

Expected value: $2(a + 2b) + 3(a + 3b) + 4(a + 4b) = 3.2$

$$a = \frac{1}{30}, \quad b = \frac{1}{10}$$

Specific behaviours

✓ indicates probabilities
✓ equation for sum of probabilities
✓ equation for expected value
✓ values of a and b

(b) Determine $\mathrm{Var}(X)$. (2 marks)

Solution

x	2	3	4
$P(X = x)$	$\frac{7}{30}$	$\frac{10}{30}$	$\frac{13}{30}$

$$\mathrm{Var}(X) = \frac{47}{75} = 0.62\overline{6}$$

Specific behaviours

✓ indicates probabilities
✓ correct variance

(c) A second random variable Y is a linear transformation of X such that $Y = kX + 4$, where k is a constant and $\mathrm{E}(Y) = 20$. Determine $\mathrm{Var}(Y)$. (2 marks)

Solution

$$3.2k + 4 = 20 \Rightarrow k = 5$$

$$\mathrm{Var}(Y) = 5^2 \times \frac{47}{75} = \frac{47}{3} = 15.\overline{6}$$

Specific behaviours

✓ indicates value of k
✓ correct variance

Question 11 **(6 marks)**

A retail chain wants to know what proportion of its customers support a recent decision to extend the range of clothes sold at its 18 stores.

(a) Comment, with reasons, on whether the following sampling methods are likely to introduce bias.

(i) Send an employee to one randomly selected store at noon on a Friday and get them to record the responses of the first 15 customers who arrive. (2 marks)

Solution
Biased, as - small sample size - only get responses of lunchtime customers - only ask users of one location, etc, etc
Specific behaviours
✓ indicates bias ✓ reason

(ii) In a newsletter sent to all customers, include a link to a public page on their website where users can click a 'yes' or 'no' button to register their support. (2 marks)

Solution
Biased, as - volunteer sampling - customers may not have internet access - web site visitors may not be customers, etc, etc
Specific behaviours
✓ indicates bias ✓ reason

(b) Following the analysis of a large random s… …orted that the 99% confidence interval for customer s… …of the statements below as **true** or **false**, where t… …ot follow logically from the report.

Examiners note
Interpretation of Confidence Intervals Suppose that a 99% confidence interval is calculated as [a, b]. A common misconception is to think this means there is a 99% chance that the true population proportion falls between a and b. This is incorrect. Like any population parameter, the population proportion is a constant, not a random variable. It does not change. The probability that a constant falls within any given range is either 0 or 1. The confidence level describes the uncertainty associated with a sampling method. Suppose we used the same sampling method to select different samples and to compute a different interval estimate for each sample. Some interval estimates would include the true population proportion, and some would not. A 99% confidence level means that we would expect 99% of the interval estimates to include the true population proportion.

(i) If the random sampling was repeate… …the true proportion of supportive customers … (1 mark)

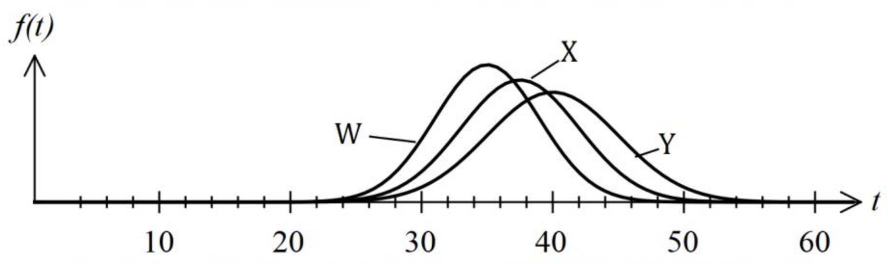
Solution
False. (See notes)
Specific behaviours
✓ correct response

(ii) There is a 99% chance that the true… …es between 0.8 and 0.9. (1 mark)

Solution
False. (See notes)
Specific behaviours
✓ correct response

Question 12 **(8 marks)**

(a) The graphs of the probability density functions of three normally distributed random variables W, X and Y are shown below.



State, with justification, which of the three random variables has

(i) the largest standard deviation? (1 mark)

Solution
Y - lowest f_{max}, so most spread.
Specific behaviours
✓ correct variable with reason

(ii) the largest mean? (1 mark)

Solution
Y - maximum furthest to right
Specific behaviours
✓ correct variable with reason

(b) Empty bottles are filled with A mL of water, where A is a normally distributed random variable with mean of 510 mL and standard deviation of 7.5 mL.

(i) Determine the probability that a bottle is filled with more than 520 mL. (1 mark)

Solution
$P(X > 520) = 0.0912$
Specific behaviours
✓ correct probability

(ii) Determine the probability that a bottle is filled with less than 515 mL, given that it is filled with more than 510 mL. (2 marks)

Solution
$P(510 < A < 515) = 0.2475$ $p = \dfrac{0.2475}{0.5} = 0.4950$
Specific behaviours
✓ numerator   ✓ correct probability

(iii) The mean of A is to be decreased by k mL so that just 2.5% of bottles are filled with 520 mL or more. Determine the value of k. (3 marks)

Solution
$\dfrac{520 - \bar{x}}{7.5} = -1.96$ $\bar{x} = 505.3$ $k = 510 - 505.3 = 4.7 \text{ mL}$
Specific behaviours
✓ equation showing correct z-score   ✓ solves for mean   ✓ correct value of k

Question 13 **(8 marks)**

235 out of a random sample of 855 people in a city had seen their dentist in the last year.

(a) If there were $219\ 000$ people living in the city, estimate the actual number of these who had seen their dentist in the last year. (2 marks)

Solution
$235 \div 855 = 0.27485$ $0.27485 \times 219000 \approx 60\ 200$ people
Specific behaviours
✓ indicates proportion ✓ estimate, to nearest 100

(b) Determine the approximate margin of error for a 99% confidence interval for the proportion of people who had seen their dentist in the last year. (2 marks)

Solution
$\text{sd} = \sqrt{\dfrac{0.27485(1-0.27485)}{855}} = 0.01527$ $E = 0.01527 \times 2.576 = 0.0393$
Specific behaviours
✓ indicates standard deviation ✓ correct margin of error

(c) Determine an approximate 99% confidence interval for the true proportion of people who had seen their dentist in the last year. (2 marks)

Solution
0.27485 ± 0.0393 $[0.2355, 0.3142]$
Specific behaviours
✓ indicates $\hat{p} \pm E$ ✓ correct interval

(d) In order to confirm the sample proportion obtained from the random sample, another sample is to be taken. Estimate, to the nearest 10 people, the sample size required to obtain an approximate margin of error for a 99% confidence interval that is close to 0.07. (2 marks)

Solution
$n = \dfrac{2.576^2(0.27485)(1-0.27485)}{0.07^2}$ $n \approx 270$
Specific behaviours
✓ indicates correct method ✓ correct size

Question 14 **(10 marks)**

Every day a fisheries researcher randomly catches 10 fish from an inland lake containing a large number of fish, 68% of which are thought to be perch.

(a) The random variable X is the number of perch in the daily catch.

(i) Describe the distribution of X. (2 marks)

Solution
$X \sim B(10, 0.68)$
Specific behaviours
✓ indicates binomial ✓ indicates parameters

(ii) Over a period of 15 days, how many times would you expect the daily catch to contain more perch than fish of other species? (2 marks)

Solution
$P(X \geq 6) = 0.8133$ $n = 0.8133 \times 15 \approx 12$ days
Specific behaviours
✓ indicates probability ✓ correct number of days

(iii) Determine the probability that a total of 19 perch are caught over two consecutive days. (2 marks)

Solution	
$p = P(X = 9) \times P(X = 10) \times 2$ $= 0.0995 \times 0.0211 \times 2 = 0.0042$	$Y \sim B(20, 0.68)$ $P(Y = 19) = 0.0042$
Specific behaviours	
✓ indicates method ✓ correct probability	

(b) The researcher suspected that the proportion of perch was lower than initially thought, but more than 60%.

(i) Calculate an approximate 90% confidence interval for the proportion of perch in the lake given that over a 7-day period, a total of 49 perch were caught. (2 marks)

Solution
$x = 49, n = 70, p = 0.7$ CI: $[0.61, 0.79]$
Specific behaviours
✓ indicates x and n ✓ states interval

(ii) Use the confidence interval to comment on the researcher's suspicion. (2 marks)

Solution
No evidence that proportion is lower, as 68% is within the CI, but there is evidence that proportion is more than 60%, as 60% is below the lower bound of CI.
Specific behaviours
✓ comment on lower than 68% ✓ comment on more than 60%

Question 15 **(7 marks)**

The table below shows the sign of the polynomial $f(x)$ and some of its derivatives at various values of x. There are no other zeroes of $f(x), f'(x)$ or $f''(x)$ apart from those shown in the table.

x	-2	-1	0	1	2	3	4
$f(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	$-$	0	$+$	$+$	0	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$	0	$+$

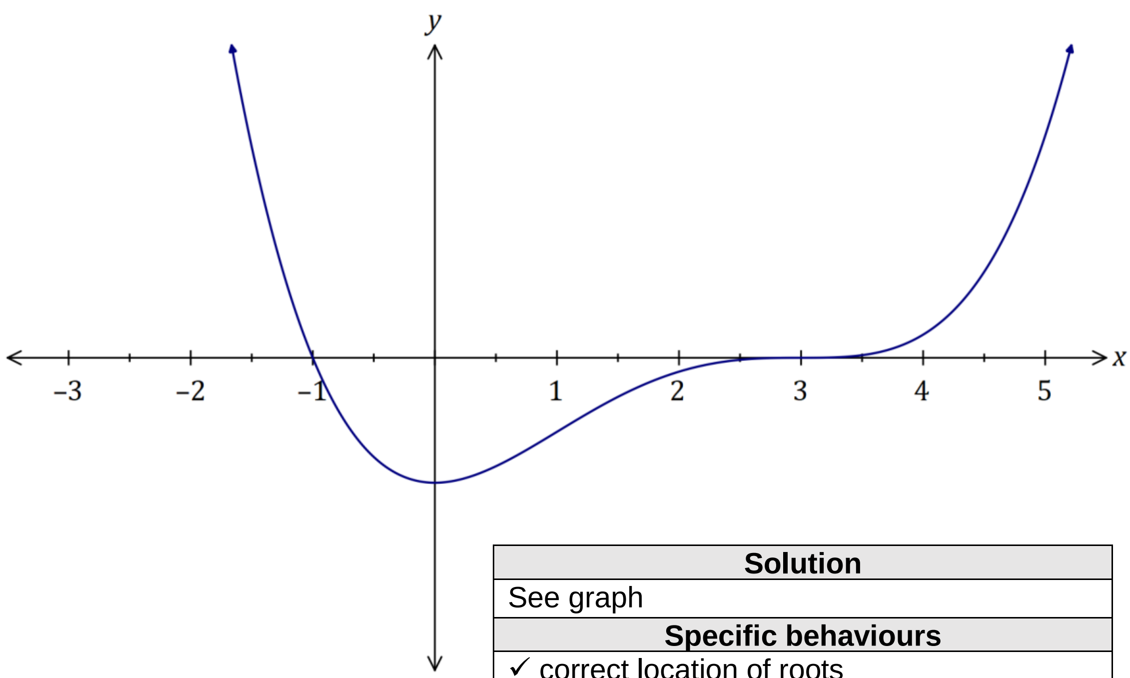
(a) For what value(s) of x is the graph of the function concave up? (1 mark)

Solution
$x < 1$ and $x > 3$
Specific behaviours
✓ correct inequalities and domain

(b) At what location does the graph of f have a turning point? Explain your answer. (2 marks)

Solution
At $x = 0$. The gradient is zero and f is concave up on either side.
Specific behaviours
✓ location ✓ explanation

(c) Sketch a possible graph of $y = f(x)$ on the axes below. (4 marks)



Solution
See graph
Specific behaviours
✓ correct location of roots ✓ correct location of stationary points ✓ point of inflection at $x = 1$ ✓ concave up everywhere except $1 < x < 3$

Question 16 **(8 marks)**

A student repeatedly took random samples of size 150 from a large population in which it was known that 38% of people were classified as overweight. For each sample, the proportion of overweight people was calculated and recorded as the sample proportion.

(a) Use an appropriate binomial distribution to determine the probability that the sample proportion is no more than 0.34 in a randomly chosen sample. (3 marks)

Solution
$X \sim B(150, 0.38)$ $0.34 \times 150 = 51$ $P(X \leq 51) = 0.1777$
Specific behaviours
✓ states parameters ✓ indicates most number of successes ✓ correct probability

(b) After recording a large number of sample proportions, the student used them to create a histogram from which the approximate normality of their distribution was evident.

(i) Determine the expected mean and standard deviation of the observed normal distribution. (2 marks)

Solution
$\text{mean} = 0.38$ $\text{sd} = \sqrt{\dfrac{0.38(1-0.38)}{150}} \approx 0.0396$
Specific behaviours
✓ correct mean ✓ correct sd

(ii) Use this normal distribution to determine the probability that the sample proportion is no more than 0.34 in a randomly chosen sample. (1 mark)

Solution
$P(X < 0.34) = 0.1564$
Specific behaviours
✓ correct probability

(iii) Describe how the parameters calculated in (i) would change if the student took smaller random samples. (2 marks)

Solution
Mean would stay the same. SD would increase.
Specific behaviours
✓ states no change in mean ✓ states increase in sd

Question 17 **(7 marks)**

At time $t = 0$, a small body P is at the origin O and is moving with a velocity of 18 ms^{-1}. The acceleration of P for $t \geq 0$ is given by

$$a = \frac{-3}{\sqrt{t+4}}\ \mathrm{ms}^{-2}.$$

(a) Determine the velocity of P when $t = 5$. (4 marks)

Solution	
$v = \int a\,dt$ $= -6\sqrt{t+4} + c$ $c = 18 + 6\sqrt{4} = 30$ $v = 30 - 6\sqrt{t+4}$ $v(5) = 30 - 6\sqrt{9} = 12\ \mathrm{ms}^{-1}$	$v(5) = 18 + \int_0^5 \left(\frac{-3}{\sqrt{t+4}}\right) dt$ $= 18 - 12 = 6$ *NB Using net change is quicker in (a), but since an expression for $v(t)$ is needed in (b) best to determine it here.*

Specific behaviours
✓ indicates v is integral of a ✓ correct integral ✓ evaluates c ✓ correct velocity

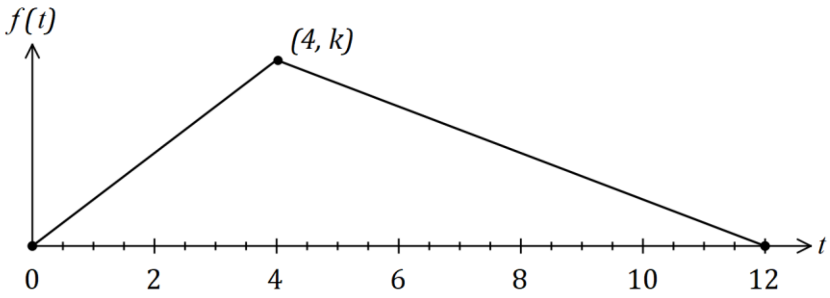
(b) Determine the distance of P from O at the instant P is stationary. (3 marks)

Solution
$v = 0 \Rightarrow 30 - 6\sqrt{t+4} = 0 \Rightarrow t = 21$ $OP = \int_0^{21} \left(30 - 6\sqrt{t+4}\right) dt$ $= 162\ \mathrm{m}$

Specific behaviours
✓ determines value of t ✓ writes integral for change in displacement ✓ correct distance

Question 18 **(11 marks)**

The time T to process orders at a warehouse is a random variable which can take any value in the interval 0 to 12 minutes. The graph of the triangular probability density function of T is shown below.



(a) Determine the value of k. (1 mark)

Solution
$\frac{1}{2}(12)(k) = 1 \Rightarrow k = \frac{1}{6}$
Specific behaviours
✓ correct value

(b) Determine the probability that the time to process an order takes less than 3 minutes. (3 marks)

Solution
$f(t) = \frac{t}{24}, 0 \leq t \leq 4$ $P(T < 3) = \int_0^3 \left(\frac{t}{24}\right) dt = \frac{3}{16} = 0.1875$
Specific behaviours
✓ indicates $f(t)$ for interval   ✓ indicates integral   ✓ correct probability

(c) Determine the mean time to process an order in minutes and seconds. (4 marks)

Solution
$g(t) = \dfrac{-1}{48}(t-12), \qquad 4 < t \le 12$ $\mathrm{E}(T) = \int_0^4 t\left(\dfrac{t}{24}\right)dt + \int_4^{12} t\left(\dfrac{-1}{48}(t-12)\right)dt$ $= \dfrac{16}{3} = 5\tfrac{1}{3}$ Mean is 5 min 20 sec.
Specific behaviours
✓ indicates $g(t)$ for second interval   ✓ indicates both integrals   ✓ evaluates mean   ✓ writes mean as required

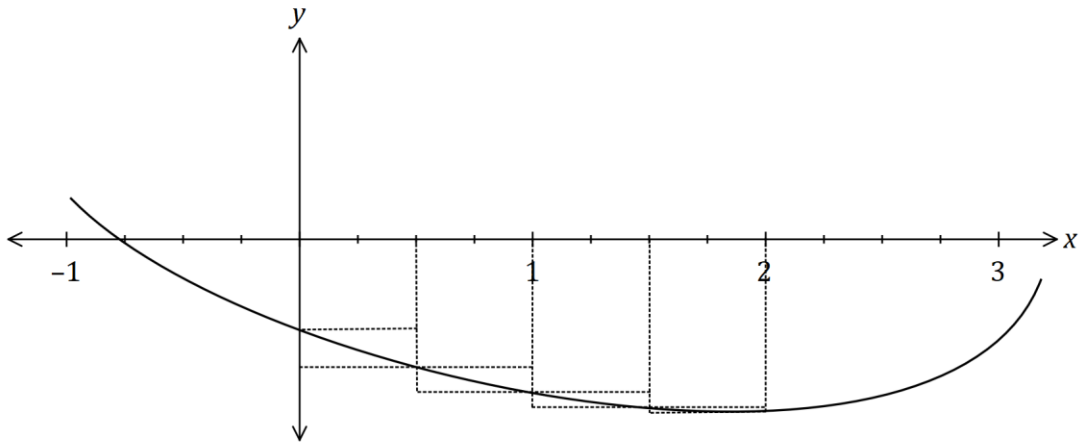
The variance of T is 6 minutes 13 seconds.

(d) Two new procedures will affect the processing time of an order. The first will decrease the time by 15% and the second will then add one-and-a-half minutes. Determine the new mean and standard deviation of the time to process an order. (3 marks)

Solution
$\mathrm{E}(0.85T + 1.5) = 0.85 \times 5\tfrac{1}{3} + 1.5$ $= 6.03 \text{ min } (6 \text{ m } 2 \text{ s})$ $\sigma_{old} = \sqrt{6\tfrac{13}{60}} = 2.493$ $\sigma_{new} = 0.85 \times 2.493$ $= 2.12 \text{ min } (2 \text{ m } 7 \text{ s})$
Specific behaviours
✓ new mean   ✓ indicates original sd   ✓ new sd

Question 19 **(9 marks)**

(a) The graph of $y = f(x)$ is shown together with some values of $f(x)$.



x	-0.5	0	0.5	1	1.5	2	2.5
$f(x)$	-3.2	-6.6	-8.6	-11.1	-11.9	-12.2	-11.5

By considering the areas of the rectangles shown and using values of $f(x)$ from the table,

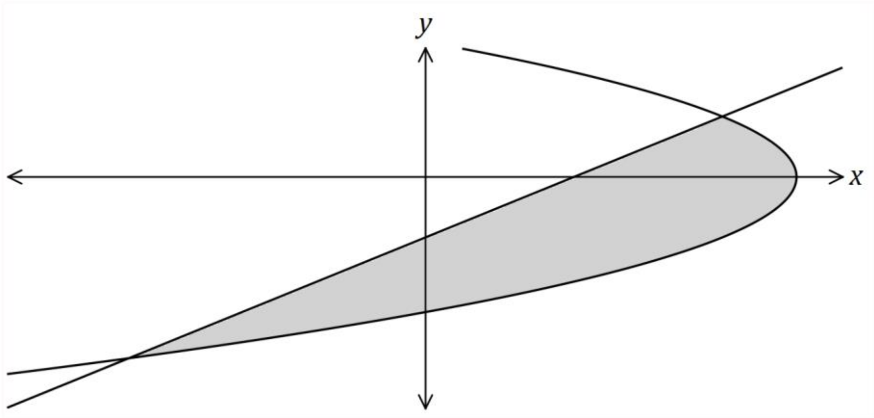
(i) calculate an underestimate for the numerical approximation for $\int_0^2 f(x)\,dx$. (2 marks)

Solution
$\text{Under estimate} = 0.5(-11.9 - 11.1 - 8.6 - 6.6)$ $= -19.1$
Specific behaviours
✓ correct values of $f(x)$ used ✓ under estimate

(ii) calculate, using rectangles, a more accurate numerical approximation for $\int_0^2 f(x)\,dx$. (3 marks)

Solution
$\text{Over estimate} = 0.5(12.2 - 11.9 - 11.1 - 8.6)$ $= -21.9$ $\text{Area estimate} = (-21.9 - 19.1) \div 2 = -20.5$ $\therefore \int_0^2 f(x)\,dx \approx -20.5$
Specific behaviours
✓ over estimate of area ✓ averages for best estimate of area ✓ correct sign for integral

(b) The graph of $x = 10 - 2y^2$ and the line $4y = x - 4$ are shown below.



Determine the area bounded by the line and the curve. (4 marks)

Solution
Line-curve intersect when $x = -8, 8$ (CAS) When $y = 0$, $x = 10$. Curve: $y = \pm\sqrt{5 - 0.5x}$ Line: $y = 0.25x - 1$ $A_1 = \int_{-8}^{8} \left((0.25x - 1) - \left(-\sqrt{5 - 0.5x}\right)\right) dx = \frac{56}{3}$ $A_2 = \int_{8}^{10} \left(\sqrt{5 - 0.5x}\right) - \left(-\sqrt{5 - 0.5x}\right) dx = \frac{8}{3}$ $A = A_1 + A_2 = \frac{64}{3}$ sq units
Specific behaviours
✓ points of intersection ✓ correct integral A_1 ✓ correct integral A_2 ✓ correct area

Alternative Solution
Line and curve intersect when $y = -3, 1$ (CAS) Line: $x = 4y + 4$ $A = \int_{-3}^{1} \left((10 - 2y^2) - (4y + 4)\right) dy = \frac{64}{3}$ sq units
Specific behaviours
✓ points of intersection ✓ correct integrand ✓ correct bounds ✓ correct area

Question 20 **(6 marks)**

A game is played at a carnival where two fair 4-sided dice with faces numbered $1, 2, 3$ and 4 are tossed at the same time. Patrons pay $\$3$ for each play of the game, winning a major prize if both dice show a four or a minor prize if just one of the dice shows a four. The operator of the game buys major prizes for $\$22$ each, minor prizes for $\$2.50$ and must pay overhead costs of $\$95$ per day.

Determine how many times the game must be played per day so that the operator can expect to make a daily profit of at least $\$150$.

Solution

$X = \$ \text{ profit per game for operator}$

x	-19.00	0.50	3.00
$P(X = x)$	$\frac{1}{16}$	$\frac{6}{16}$	$\frac{9}{16}$

$$\mathrm{E}(X) = \frac{-19 + 3 + 27}{16} = \frac{11}{16} = \$0.6875$$

$$\frac{11}{16}n \geq 95 + 150$$
$$n \geq 356.4$$

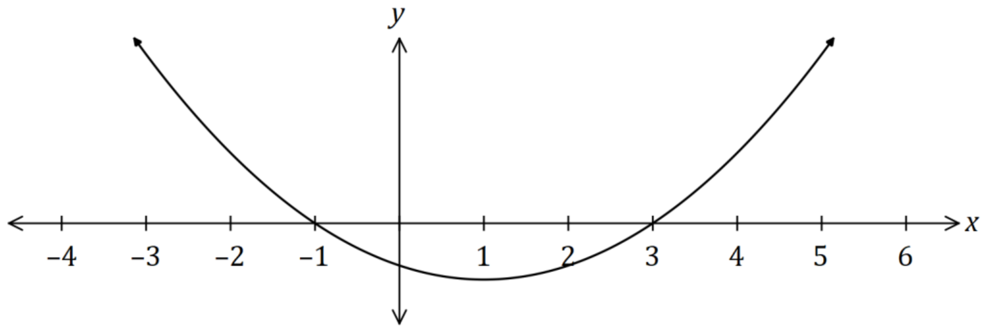
Require at least 357 patrons to play per day.

Specific behaviours

- ✓ defines random variable
- ✓ table with row showing values RV can take
- ✓ correct probabilities for all outcomes
- ✓ calculates $\mathrm{E}(X)$
- ✓ forms inequality $n \times \mathrm{E}(X) \geq \text{overheads+profit}$
- ✓ solves inequality and writes solution

Question 21 **(4 marks)**

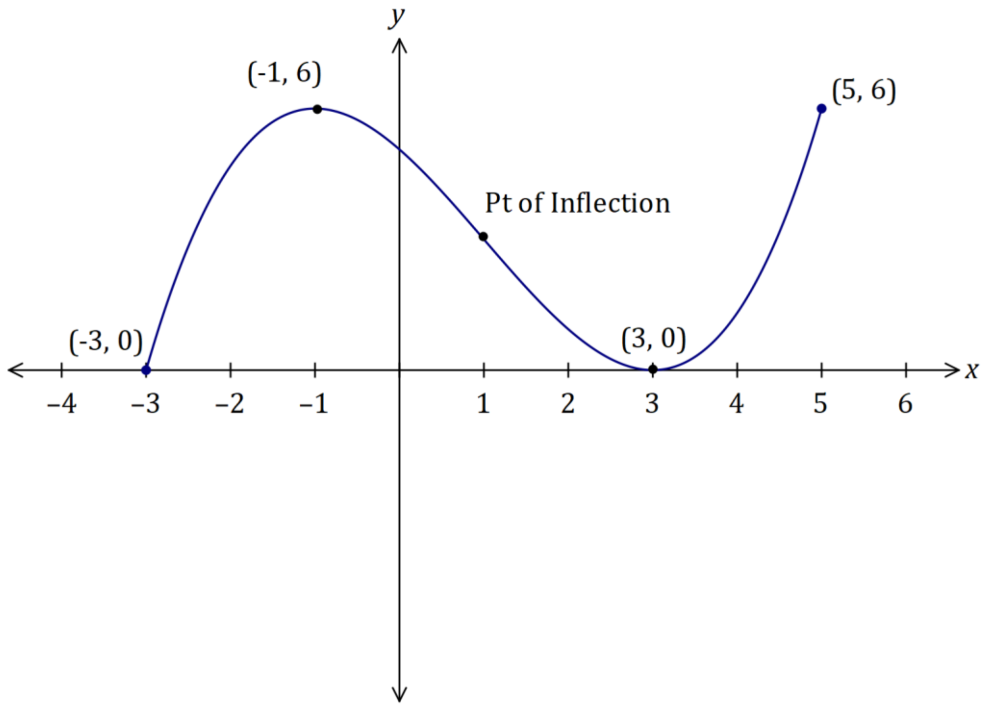
The graph of $y = f(x)$ is shown below.



Another function A is defined on the interval $-3 \le x \le 5$ by

$$A(x) = \int_{-3}^{x} f(t)\,dt.$$

It is known that $A(-1) = A(5) = 6$ and $A(3) = 0$. Sketch the graph of $y = A(x)$ on the axes below, clearly indicating the location of all x-intercepts, turning points, points of inflection and other key features.



Solution
See graph
Specific behaviours
✓ sketched over defined interval ✓ x-intercepts ✓ local minimum ✓ labelled endpoint

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________